University of KwaZulu-Natal

School of Engineering

Electrical, Electronic & Computer Engineering

MAIN EXAMINATIONS – NOVEMBER 2015

ENEL2FT: FIELD THEORY

Time allowed: 2 hours

Instructions to Candidates:

1. This paper contains 4 questions
2. Answer any **THREE** questions.
3. All questions carry equal marks. The marks for each question/section are indicated.
4. Answers should show sufficient working steps to indicate the solution method used.
5. Any additional examination material is to be placed in the answer book and must indicate clearly the question number and the Student Registration number

The following materials are provided:

1. Graph Paper
2. Mathematical Tables & Formula Sheet

Examiners

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Durban, November 2015

Question 1 (25 Marks)

a) Four 10nC positive charges are located in the z = 0 plane at the corners of a square of length 8 cm on a side. A fifth 10nC positive charge is located at a point 8 cm distant from the other charges. Calculate the magnitude of the total force on this fifth charge for $\varepsilon=\varepsilon_o$.

(7 marks)

b) A uniform volume charge density ρ_v=0.2 μC/m^3 is present throughout the spherical shell extending from r = 3 cm to r = 5 cm. If ρ_v = 0 elsewhere, determine:

i) The total charge present throughout the shell

ii) The radius r_1 if half the total charge is located in the region 3cm<r<r_1.

(8 marks)

c) A circular disk of radius ρ=4 m, lying on the z=0 plane, centred at z=0, ρ=0, has surface charge density given by:

$$\rho_s = \frac{10^{-4}}{\rho} \quad C/m^2$$

Determine, the electric field intensity $\vec{E}$ at point P which is located at ρ=0, z=3 m.

(10 marks)

Question 2 (25 Marks)

a) The surface x=0 separates two perfect dielectrics. For x>0, dielectric material 2 has relative permittivity ε_{r2}=2.4, while for x<0, material 1 has ε_{r1}=1. If the electric flux density in region 1 is given by $\vec{D}_1 = 3\hat{x} - 4\hat{y} + 6\hat{z}\ C/m^2$ determine:

   i) $\vec{E}$ and $\vec{D}$ in dielectric material 2.

   ii) The angles that the $\vec{E}$ vectors make with the tangent in both media.

(7 marks)

b) The electric flux density in a certain region is given (in cylindrical coordinates) by:

$$\vec{D} = 8\rho \sin\phi \hat{\rho} + 4\rho \cos\phi \hat{\phi}\ C/m^2$$

Determine:

   i) The volume charge density, and evaluate it at P(2.6 m, 38°, -6.1 m)

   ii) How much charge is located in the region defined by: $0 \le \rho \le 1.8$ m; $20^o \le \phi \le 70^o$; $2.4 \le z \le 3.1$ m.

(7 marks)

c) Let a filamentary current of 5 mA be directed from infinity to the origin on the positive *z* axis and then back out to infinity on the positive *x* axis. Find $\vec{H}$ at *P*(0, 1, 0)

(11 marks)

Question 3 (25 Marks)

a) Two concentric spheres have a dielectric material of relative permittivity ε_r=3.12 placed in the region between them. The inner sphere has radius r=2 cm, and is placed at a potential V_1=-25 volts; while the outer sphere has radius r=35 cm, and is at a potential V_2=150 volts.

   i) Using Laplace's equations, solve for potential V in the region between the spheres
   ii) Determine the electric field intensity, $\vec{E}$ in the region between the spheres.
   iii) Find the surface charge on each sphere.
   iv) Determine the capacitance between the spheres

(13 marks)

b) A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2A dc. Determine:

   i) The current density $\vec{J}$ within the conductor
   ii) The magnetic field intensity $\vec{H}$ within the conductor. Use Ampere's circuital law
   iii) That $\vec{J} = \vec{\nabla} X \vec{H}$ inside the conductor
   iv) The magnetic field intensity $\vec{H}$ outside the conductor. Use Ampere's circuital law
   v) That $\vec{J} = \vec{\nabla} X \vec{H}$ outside the conductor

(12 marks)

Question 4 (25 Marks)

a) In a region with cylindrical symmetry, the conductivity, $\sigma = 1500e^{-150\rho}\ S/m$. If an electrostatic field of $\vec{E} = 30\hat{z}\ V/m$, is present, determine:

i) The expression for the current density, $\vec{J}$.

ii) The total current leaving the surface defined by $\rho<\infty$, z=0, $0<\phi<2\pi$.

iii) The magnetic field intensity, $\vec{H}$.

(8 marks)

b) A surface current sheet $\vec{K} = 9\hat{y}\ A/m$ is located in the plane z=0, the interface between region 1, z<0, with relative permeability μ_r=4; and region 2, z>0, with μ_r =3. If $\vec{H}_2 = 14.5\hat{x} + 8.0\hat{z}\ A/m$ determine $\vec{H}_1$.

(8 marks)

c) A torroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm^2 and an effective length of 8 cm. There is also a short air gap of 0.25 mm length and an effective area of 2.8 cm^2. An mmf of 200 Ampere-turns is applied to the magnetic circuit. Calculate the total flux in the torroid if:

i) The magnetic material is assumed to have infinite permeability

ii) The magnetic material is assumed to be linear with relative permeability μ_r=1000.

(9 marks)

******************************END OF PAPER*****************************

VECTOR ANALYSIS

Coordinate Transformations

Rectangular to cylindrical:

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\rho}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

Rectangular to spherical:

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0

Cylindrical to spherical:

	$\hat{\rho}$	$\hat{\phi}$	$\hat{z}$
$\hat{r}$	$\sin\theta$	0	$\cos\theta$
$\hat{\theta}$	$\cos\theta$	0	$-\sin\theta$
$\hat{\phi}$	0	1	0

These tables can be used to transform unit vectors as well as vector components; e.g.,

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$A_\rho = A_x\cos\phi + A_y\sin\phi$$

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Vector Differential Operators

Rectangular coordinates:

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\bar{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2\bar{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$$

Cylindrical coordinates:

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

$$\nabla\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\bar{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right]$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla\times\nabla\times\bar{A}$$

Spherical coordinates:

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\bar{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\nabla^2\bar{A} = \nabla\nabla\cdot\bar{A} - \nabla\times\nabla\times\bar{A}$$

VECTOR FORMULAS

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \qquad (1\text{-}29)$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad (1\text{-}30)$$

$$\nabla\times\nabla u = 0 \qquad (1\text{-}48)$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \qquad (1\text{-}49)$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \qquad (1\text{-}106)$$

$$\frac{d}{d\sigma}(u\mathbf{A}) = \frac{du}{d\sigma}\mathbf{A} + u\frac{d\mathbf{A}}{d\sigma} \qquad (1\text{-}107)$$

$$\frac{d}{d\sigma}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\cdot\mathbf{B} + \mathbf{A}\ \frac{d\mathbf{B}}{d\sigma} \qquad (1\text{-}108)$$

$$\frac{d}{d\sigma}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{d\sigma} \qquad (1\text{-}109)$$

$$\nabla(u+v) = \nabla u + \nabla v \qquad (1\text{-}110)$$

$$\nabla(uv) = u\nabla v + v\nabla u \qquad (1\text{-}111)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} \qquad (1\text{-}112)$$

$$\nabla(\mathbf{C}\cdot\mathbf{r}) = \mathbf{C} \qquad \text{where } \mathbf{C} = \text{const.} \qquad (1\text{-}113)$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \qquad (1\text{-}114)$$

$$\nabla\cdot(u\mathbf{A}) = \mathbf{A}\cdot(\nabla u) + u(\nabla\cdot\mathbf{A}) \qquad (1\text{-}115)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \qquad (1\text{-}116)$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \qquad (1\text{-}117)$$

$$\nabla\times(u\mathbf{A}) = (\nabla u)\times\mathbf{A} + u(\nabla\times\mathbf{A}) \qquad (1\text{-}118)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \qquad (1\text{-}119)$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \qquad (1\text{-}120)$$

where

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\mathbf{B} = {}&\hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\\ &+\hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\\ &+\hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\end{aligned} \qquad (1\text{-}121)$$

VECTOR OPERATIONS

RECTANGULAR COORDINATES

$$\nabla u = \hat{\mathbf{x}}\frac{\partial u}{\partial x} + \hat{\mathbf{y}}\frac{\partial u}{\partial y} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \qquad (1\text{-}37)$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (1\text{-}42)$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad (1\text{-}43)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad (1\text{-}46)$$

CYLINDRICAL COORDINATES

$$\nabla u = \hat{\boldsymbol{\rho}}\frac{\partial u}{\partial\rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial u}{\partial\varphi} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \qquad (1\text{-}85)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \qquad (1\text{-}87)$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right] \qquad (1\text{-}88)$$

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} \qquad (1\text{-}89)$$

SPHERICAL COORDINATES

$$\nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} \qquad (1\text{-}101)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \qquad (1\text{-}103)$$

$$\begin{aligned}\nabla\times\mathbf{A} = {}&\frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right]\\ &+\frac{\hat{\boldsymbol{\varphi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \qquad (1\text{-}104)$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} \qquad (1\text{-}105)$$

Useful mathematical tables

C.1 A brief list of series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$
$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$
$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots |x| < 1$$
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots = \infty$$
$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln(2)$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$
$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \qquad \text{for all } x$$

.2 A list of trigonometric identities

$$e^{\theta} = \cosh(\theta) + \sinh(\theta) = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \text{where } j = \sqrt{-1}$$
$$\cosh(\theta) = \tfrac{1}{2}[e^{\theta} + e^{-\theta}]$$

$$\sinh(\theta) = \tfrac{1}{2}[e^{\theta} - e^{-\theta}]$$
$$\cos(\theta) = \tfrac{1}{2}[e^{j\theta} + e^{-j\theta}]$$
$$\sin(\theta) = \tfrac{1}{2j}[e^{j\theta} - e^{-j\theta}]$$
$$\sin(-\alpha) = -\sin(\alpha) \qquad \sin(\alpha) = \cos(\alpha - \pi/2)$$
$$\cos(-\alpha) = \cos(\alpha) \qquad \cos(\alpha) = -\sin(\alpha - \pi/2)$$
$$\cosh(j\alpha) = \cos(\alpha)$$
$$\sinh(j\alpha) = j\sin(\alpha)$$
$$\cos(j\beta) = \cosh(\beta)$$
$$\sin(j\beta) = j\sinh(\beta)$$
$$\sinh(\alpha + \beta) = \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$$
$$\cosh(\alpha + \beta) = \cosh(\alpha)\cosh(\beta) + \sinh(\alpha)\sinh(\beta)$$
$$\sinh(\alpha + j\beta) = \sinh(\alpha)\cos(\beta) + j\cosh(\alpha)\sin(\beta)$$
$$\cosh(\alpha + j\beta) = \cosh(\alpha)\cos(\beta) + j\sinh(\alpha)\sin(\beta)$$
$$\sin(\alpha + j\beta) = \sin(\alpha)\cosh(\beta) + j\cos(\alpha)\sinh(\beta)$$
$$\sin(\alpha - j\beta) = \sin(\alpha)\cosh(\beta) - j\cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + j\beta) = \cos(\alpha)\cosh(\beta) - j\sin(\alpha)\sinh(\beta)$$
$$\cos(\alpha - j\beta) = \cos(\alpha)\cosh(\beta) + j\sin(\alpha)\sinh(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$
$$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$
$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)\\ &= 2\cos^2(\alpha) - 1\\ &= 1 - 2\sin^2(\alpha)\end{aligned}$$
$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$$
$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$1 + \tan^2(\alpha) = \sec^2(\alpha) \qquad 1 + \cot^2(\alpha) = \csc^2(\alpha)$$
$$\sin^2(\alpha) = \tfrac{1}{2}(1 - \cos(2\alpha))$$
$$\cos^2(\alpha) = \tfrac{1}{2}(1 + \cos(2\alpha))$$
$$\sin^3(\alpha) = \tfrac{1}{4}(3\sin(\alpha) - \sin(3\alpha))$$
$$\cos^3(\alpha) = \tfrac{1}{4}(3\cos(\alpha) + \cos(3\alpha))$$
$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$
$$2\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$
$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$
$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

C.3 A list of indefinite integrals

In the list of integrals that follows, C is simply a constant of integration.

Let $X = \sqrt{a^2 + x^2}$

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$$

$$\int X\,dx = \frac{1}{2}x\,X + \frac{a^2}{2}\ln|x + X| + C$$

$$\int xX\,dx = \frac{1}{3}X^3 + C$$

$$\int \frac{dx}{X} = \ln[x + X] + C$$

$$\int \frac{dx}{X^3} = \frac{1}{a^2}\frac{x}{X} + C$$

$$\int \frac{dx}{X^5} = \frac{1}{a^4}\left[\frac{x}{X} - \frac{1}{3}\frac{x^3}{X^3}\right] + C$$

$$\int \frac{x\,dx}{X} = X + C$$

$$\int \frac{x\,dx}{X^3} = -\frac{1}{X} + C$$

$$\int \frac{x\,dx}{X^5} = -\frac{1}{3X^3} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$$

$$\int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3}\tan^{-1}(x/a) + C$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln|a^2 + x^2| + C$$

$$\int \frac{x\,dx}{(a^2 + x^2)^2} = -\frac{1}{2(a^2 + x^2)} + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln|(a + x)/(a - x)| + C = \frac{1}{a}\tanh^{-1}(x/a) + C$$

$$\int \frac{x\,dx}{(a^2 - x^2)} = -\frac{1}{2}\ln|a^2 - x^2| + C$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$$

Appendix C Useful mathematical tables

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\cos(ax)\,dx = -\frac{\cos(2ax)}{4a} + C$$

$$\int \sin^n(ax)\cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{(n+1)a} + C, \quad n \neq -1$$

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln|\cos(ax)| + C$$

$$\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x + C$$

$$\int \cot^2(ax)\,dx = -\frac{1}{a}\cot(ax) - x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int b^{ax}\,dx = \frac{1}{a\ln(b)}b^{ax} + C$$

$$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$$

$$\int \ln(ax)\,dx = x\ln(ax) - x + C$$

$$\int x^n\ln(ax)\,dx = \frac{x^{n+1}}{n+1}\ln(ax) - \frac{x^{n+1}}{(n+1)^2} + C \qquad n \neq -1$$

$$\int \frac{1}{x}\ln(ax)\,dx = \frac{1}{2}[\ln(ax)]^2 + C$$

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax) + C$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax) + C$$

$$\int \tanh(ax)\,dx = \frac{1}{a}\ln[\cosh(ax)] + C$$

$$\int \coth(ax)\,dx = \frac{1}{a}\ln|\sinh(ax)| + C$$

$$\int \operatorname{sech}(ax)\,dx = \frac{1}{a}\sin^{-1}[\tanh(ax)] + C$$

$$\int \operatorname{csch}(ax)\,dx = \frac{1}{a}\ln|\tanh(ax/2)| + C$$

$$\int \sinh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} - \frac{x}{2} + C$$

$$\int \cosh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} + \frac{x}{2} + C$$

$$\int \tanh^2(ax) = x - \frac{1}{a}\tanh(ax) + C$$

$$\int \coth^2(ax)\,dx = x - \frac{1}{a}\coth(ax) + C$$

$$\int \operatorname{sech}^2(ax)\,dx = \frac{1}{a}\tanh(ax) + C$$

$$\int \operatorname{csch}^2(ax)\,dx = -\frac{1}{a}\coth(ax) + C$$

C.4 A partial list of definite integrals

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \qquad (a > 0)$$

$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2} \qquad (a > 0)$$

$$\int_0^\infty x^2e^{-ax}\,dx = \frac{2}{a^3} \qquad (a > 0)$$

$$\int_0^\infty x^ne^{-ax}\,dx = \frac{n!}{a^{n+1}} \qquad (a > 0, n > -1)$$

$$\int_0^\infty x^{1/2}e^{-ax}\,dx = \frac{1}{2a}\sqrt{\pi/a} \qquad (a > 0)$$

$$\int_0^\infty x^{-1/2}e^{-ax}\,dx = \sqrt{\pi/a} \qquad (a > 0)$$

$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2} \qquad (a > 0)$$

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2} \qquad (a > 0)$$

$$\int_0^\infty xe^{-ax}\sin(bx)\,dx = \frac{2ab}{(a^2+b^2)^2} \qquad (a > 0)$$

Appendix C Useful mathematical tables

$$\int_0^{\infty} xe^{-ax}\cos(bx)\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \qquad (a>0)$$

$$\int_0^{2\pi} \sin(ax)\,dx = 0 \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{2\pi} \cos(ax)\,dx = 0 \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{2\pi} \sin^2(ax)\,dx = \pi \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{2\pi} \cos^2(ax)\,dx = \pi \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi} \cos(ax)\,dx = 0 \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi} \sin(ax)\,dx = \frac{1}{a}\left[1-\cos(a\pi)\right] \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi} \sin^2(ax)\,dx = \frac{\pi}{2} \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi} \cos^2(ax)\,dx = \frac{\pi}{2} \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi} \sin(ax)\sin(bx) = 0 \qquad a \neq b \text{ (} a \text{ and } b \text{ are integers)}$$

$$\int_0^{\pi} \cos(ax)\cos(bx) = 0 \qquad a \neq b \text{ (} a \text{ and } b \text{ are integers)}$$

$$\begin{aligned}\int_0^{\pi} \sin(ax)\cos(bx) &= 0 \qquad a = b \text{ (} a \text{ and } b \text{ are integers)}\\ &= 0 \qquad a \neq b \text{ but } (a+b) \text{ even}\\ &= \frac{2a}{a^2-b^2} \qquad a \neq b \text{ but } (a+b) \text{ odd}\end{aligned}$$

$$\int_0^{\pi/2} \sin(ax)\,dx = \frac{1}{a}\left[1-\cos(a\pi/2)\right]$$

$$\int_0^{\pi/2} \cos(ax)\,dx = \frac{1}{a}\sin(a\pi/2)$$

$$\int_0^{\pi/2} \sin^2(ax)\,dx = \frac{\pi}{4} \qquad (a = 1, 2, 3, \ldots)$$

$$\int_0^{\pi/2} \cos^2(ax)\,dx = \frac{\pi}{4} \qquad (a = 1, 2, 3, \ldots)$$

C.6 Some exact and approximate expressions for TEM waves in lossy media

	Exact	For good dielectric $\frac{\sigma}{\omega\varepsilon} << 1$	For good Conductor $\frac{\sigma}{\omega\varepsilon} >> 1$
Attenuation constant (Np/m)	$\alpha = Re\left(j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}\right)$	$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$
Phase constant (rad/m)	$\beta = Im\left(j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}\right)$	$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$
Intrinsic impedance (Ω)	$\hat{\eta} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\eta = \sqrt{\frac{\mu}{\varepsilon}}$	$\hat{\eta} = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}}$
Wavelength (m)	$\lambda = \frac{2\pi}{\beta}$	$\lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\lambda = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Wave velocity	$u_p = \frac{\omega}{\beta}$	$u_p = \frac{1}{\sqrt{\mu\varepsilon}}$	$u_p = \sqrt{\frac{2\omega}{\mu\sigma}}$
Skin depth (m)	$\delta_c = \frac{1}{\alpha}$	$\delta_c = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	$\delta_c = \sqrt{\frac{2}{\omega\mu\sigma}}$

C.7 Some physical constants

Constant	Symbol	Value
Velocity of light in vacuum	c	2.988×10^{8} m / s
Electronic charge (magnitude)	$\lvert e\rvert$	1.602×10^{-19} C
Electronic mass	m	9.109×10^{-31} kg
Electronic charge to mass ratio	$\lvert e\rvert/m$	1.759×10^{11} C / kg
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H / m
Permittivity of free space	ε_0	$8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$ F / m
Electron volt (energy)	$\lvert e\rvert V$	1.602×10^{-19} J
Boltzmann constant	k	1.381×10^{-23} J / K
Planck's constant	H	6.626×10^{-34} J. s