UNIVERSITY OF KWAZULU-NATAL

University of KwaZulu-Natal

School of Engineering

Electrical, Electronic & Computer Engineering

MAIN EXAMINATIONS – DECEMBER 2016

ENEL2FT: FIELD THEORY

Time allowed: 2 hours

Instructions to Candidates:

1. This paper contains 5 questions
2. Answer any **FOUR** questions.
3. All questions carry equal marks. The marks for each question/section are indicated.
4. Answers should show sufficient working steps to indicate the solution method used.
5. Any additional examination material is to be placed in the answer book and must indicate clearly the question number and the Student Registration number

The following materials are provided:

1. Graph Paper
2. Mathematical Tables & Formula Sheet

Examiners

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Durban, December 2016

Question 1 (25 Marks)

a) Two charges of $1\mu C$ and -$2\mu C$ are placed at $A(15, -10, -15)$ and $B(10, -20,10)$, respectively. Determine electric field intensity E at

(i) $P(0, 0, 0)$ and

(ii) $Q(20, 10, 30)$.

[16 marks]

b) The volume in spherical coordinates between $r = 3m$ and $r = 6m$ contains a uniform charge density $\rho\ C/m^3$. Use Gauss law to find electric flux density D in all regions.

[9 marks]

Question 2(25 Marks)

a) Given the electric field intensity $E = 2x^2 a_x - 4y a_y (V/m)$, find the work done in moving a point charge of $4C$

(i) from (3,0,0) to $(0, 0, 0)$ and then from (0,0,0) to $(0, 3, 0)$

(ii) from $(3, 0, 0)$ to $(0, 3, 0)$ along the straight line path joining the two points.

[12 marks]

b) Three point charges $Q_1 = 4\ mC$, Q_2 =$1mC$, and $Q_3 = -2\ mC$ are located at $(1, 1, 2)$, $(3, -1, 1)$, and $(4, 3, -2)$, respectively.

(i) Find the potential V_P at $P(1, 1, -2)$.

(ii) Calculate the potential difference V_{PQ} if Q is $(2, 2, 1)$.

[13 marks]

Question 3(25 Marks)

a) A perfectly conducting infinite plate is located in free space at $z = 0$ and a uniform infinite line charge of $30\ nC/m$ lies along the line $x = 0$ and $z = 3$. Let $V = 0$ at the conducting plate. At $P(2\ ,5, 0)$, find electric field intensity E.

[10 marks]

b) A parallel-plate capacitor has its plates at *x* = 0, *d* and the space between the plates is filled with an inhomogeneous material with permittivity $\varepsilon = \varepsilon_0\left(1 + \frac{x}{d}\right)$. If the plate at $x = d$ is maintained at V_0 while the plate at $x = 0$ is grounded, find:

(i) potential V and electric field intensity E

(ii) polarization P

(iii) surface charge density ρ_{ps} at $x = 0, d$

[15 marks]

Question 4(25 Marks)

(a) Consider the two-wire transmission line whose cross section is illustrated in Fig Q4(a). Each wire is of radius $2\ cm$ and the wires are separated $10\ cm$. The wire centered at $(0, 0)$ carries current $3\ A$ while the other centered at $(10\ cm, 0)$ carries the return current. Find the magnetic field intensity H at

(i) $(3\ cm, 0)$

(ii) $(10\ cm, 5\ cm)$

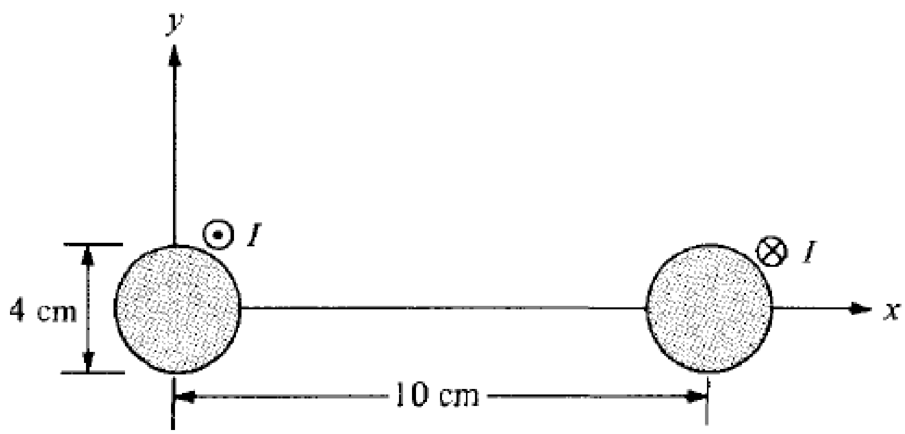

Fig Q4(a)

[10 marks]

(b) The cylindrical shell defined by $1\ cm\ < \rho < 1.4\ cm$ consists of a nonmagnetic conducting material and carries a total current of $50\ A$ in the a_z direction. Find the total magnetic flux crossing the plate $\emptyset = 0, 0 < z < 1$:

(i) $0 < \rho < 1.2\ cm$

(ii) $1\ cm < \rho < 1.4\ cm$

(iii) $1.4\ cm < \rho < 20\ cm$

[15 marks]

Question 5 (25 Marks)

a) Suppose that permittivity of region 1 is $\mu_1 = 8\ \mu H/m$ where $z > 0$, and the permittivity of region 2 is $\mu_2 = 3\ \mu H/m$ where $z < 0$. If there is a surface current density $80a_x\ A/m$ on the surface $z = 0$, and if $B_1 = 2a_x - 3a_y + a_z\ mT$ in region 1, find the value of B_2 in region 2.

[10 marks]

b) The parallel magnetic circuit shown in Fig. Q5(b) is with the same cross-sectional area throughout, $S = 1.30\ cm^2$. The mean lengths are $l_1 = l_3 = 25\ cm$, $l_2 = 5\ cm$. The coils have 50 turns each. Given that $\emptyset_1 = 90\ \mu Wb$, $\emptyset_3 = 120\ \mu Wb$, $\mu_{r1} = 6314$, $\mu_{r2} = 3730$, and $\mu_{r3} = 5230$, find the coil currents.

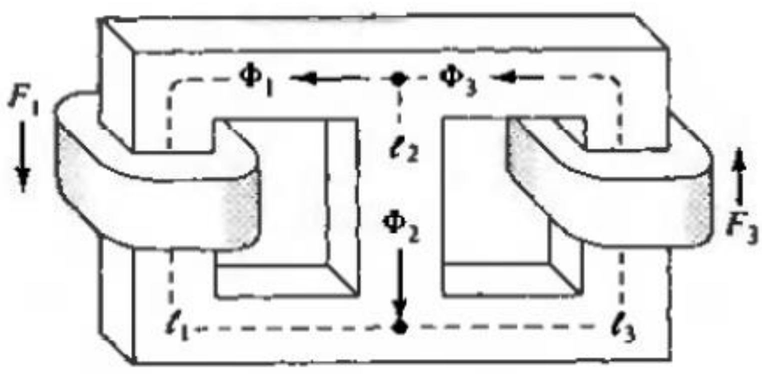

Fig. Q5(b)

[15 marks]

******************************END OF PAPER*****************************

VECTOR ANALYSIS

Coordinate Transformations

Rectangular to cylindrical:

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\rho}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{z}$	0	0	1

Rectangular to spherical:

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{r}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0

Cylindrical to spherical:

	$\hat{\rho}$	$\hat{\phi}$	$\hat{z}$
$\hat{r}$	$\sin\theta$	0	$\cos\theta$
$\hat{\theta}$	$\cos\theta$	0	$-\sin\theta$
$\hat{\phi}$	0	1	0

These tables can be used to transform unit vectors as well as vector components; e.g.,

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$A_\rho = A_x\cos\phi + A_y\sin\phi$$

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Vector Differential Operators

Rectangular coordinates:

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \bar{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \bar{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$$

Cylindrical coordinates:

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \bar{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \bar{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla \times \nabla \times \bar{A}$$

Spherical coordinates:

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \bar{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$$\nabla^2 \bar{A} = \nabla\nabla \cdot \bar{A} - \nabla \times \nabla \times \bar{A}$$

VECTOR FORMULAS

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \qquad (1\text{-}29)$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad (1\text{-}30)$$

$$\nabla\times\nabla u = 0 \qquad (1\text{-}48)$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \qquad (1\text{-}49)$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \qquad (1\text{-}106)$$

$$\frac{d}{d\sigma}(u\mathbf{A}) = \frac{du}{d\sigma}\mathbf{A} + u\frac{d\mathbf{A}}{d\sigma} \qquad (1\text{-}107)$$

$$\frac{d}{d\sigma}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\cdot\mathbf{B} + \mathbf{A}\ \frac{d\mathbf{B}}{d\sigma} \qquad (1\text{-}108)$$

$$\frac{d}{d\sigma}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{d\sigma}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{d\sigma} \qquad (1\text{-}109)$$

$$\nabla(u+v) = \nabla u + \nabla v \qquad (1\text{-}110)$$

$$\nabla(uv) = u\nabla v + v\nabla u \qquad (1\text{-}111)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} \qquad (1\text{-}112)$$

$$\nabla(\mathbf{C}\cdot\mathbf{r}) = \mathbf{C} \quad \text{where } \mathbf{C} = \text{const.} \qquad (1\text{-}113)$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \qquad (1\text{-}114)$$

$$\nabla\cdot(u\mathbf{A}) = \mathbf{A}\cdot(\nabla u) + u(\nabla\cdot\mathbf{A}) \qquad (1\text{-}115)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \qquad (1\text{-}116)$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \qquad (1\text{-}117)$$

$$\nabla\times(u\mathbf{A}) = (\nabla u)\times\mathbf{A} + u(\nabla\times\mathbf{A}) \qquad (1\text{-}118)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \qquad (1\text{-}119)$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \qquad (1\text{-}120)$$

where

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right) \qquad (1\text{-}121)$$

VECTOR OPERATIONS

RECTANGULAR COORDINATES

$$\nabla u = \hat{\mathbf{x}}\frac{\partial u}{\partial x} + \hat{\mathbf{y}}\frac{\partial u}{\partial y} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \qquad (1\text{-}37)$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (1\text{-}42)$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad (1\text{-}43)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad (1\text{-}46)$$

CYLINDRICAL COORDINATES

$$\nabla u = \hat{\boldsymbol{\rho}}\frac{\partial u}{\partial \rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial u}{\partial z} \qquad (1\text{-}85)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \qquad (1\text{-}87)$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right] \qquad (1\text{-}88)$$

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} \qquad (1\text{-}89)$$

SPHERICAL COORDINATES

$$\nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} \qquad (1\text{-}101)$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \qquad (1\text{-}103)$$

$$\nabla\times\mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right] + \frac{\hat{\boldsymbol{\varphi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \qquad (1\text{-}104)$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} \qquad (1\text{-}105)$$

Useful mathematical tables

C.1 A brief list of series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$
$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots |x| < 1$$
$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots |x| < 1$$
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots = \infty$$
$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln(2)$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$
$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \qquad \text{for all } x$$

C.2 A list of trigonometric identities

$$e^{\theta} = \cosh(\theta) + \sinh(\theta) = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \text{where } j = \sqrt{-1}$$
$$\cosh(\theta) = \tfrac{1}{2}[e^{\theta} + e^{-\theta}]$$

$\sinh(\theta) = \frac{1}{2}[e^{\theta} - e^{-\theta}]$

$\cos(\theta) = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$

$\sin(\theta) = \frac{1}{2j}[e^{j\theta} - e^{-j\theta}]$

$\sin(-\alpha) = -\sin(\alpha) \qquad \sin(\alpha) = \cos(\alpha - \pi/2)$

$\cos(-\alpha) = \cos(\alpha) \qquad \cos(\alpha) = -\sin(\alpha - \pi/2)$

$\cosh(j\alpha) = \cos(\alpha)$

$\sinh(j\alpha) = j\sin(\alpha)$

$\cos(j\beta) = \cosh(\beta)$

$\sin(j\beta) = j\sinh(\beta)$

$\sinh(\alpha + \beta) = \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$

$\cosh(\alpha + \beta) = \cosh(\alpha)\cosh(\beta) + \sinh(\alpha)\sinh(\beta)$

$\sinh(\alpha + j\beta) = \sinh(\alpha)\cos(\beta) + j\cosh(\alpha)\sin(\beta)$

$\cosh(\alpha + j\beta) = \cosh(\alpha)\cos(\beta) + j\sinh(\alpha)\sin(\beta)$

$\sin(\alpha + j\beta) = \sin(\alpha)\cosh(\beta) + j\cos(\alpha)\sinh(\beta)$

$\sin(\alpha - j\beta) = \sin(\alpha)\cosh(\beta) - j\cos(\alpha)\sin(\beta)$

$\cos(\alpha + j\beta) = \cos(\alpha)\cosh(\beta) - j\sin(\alpha)\sinh(\beta)$

$\cos(\alpha - j\beta) = \cos(\alpha)\cosh(\beta) + j\sin(\alpha)\sinh(\beta)$

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$

$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)\\ &= 2\cos^2(\alpha) - 1\\ &= 1 - 2\sin^2(\alpha)\end{aligned}$$

$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$

$\sin^2(\alpha) + \cos^2(\alpha) = 1$

$1 + \tan^2(\alpha) = \sec^2(\alpha) \qquad 1 + \cot^2(\alpha) = \csc^2(\alpha)$

$\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$

$\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$

$\sin^3(\alpha) = \frac{1}{4}(3\sin(\alpha) - \sin(3\alpha))$

$\cos^3(\alpha) = \frac{1}{4}(3\cos(\alpha) + \cos(3\alpha))$

$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

$2\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

C.3 A list of indefinite integrals

In the list of integrals that follows, C is simply a constant of integration.

Let $X = \sqrt{a^2 + x^2}$

$$\int x^{1/2}\, dx = \frac{2}{3} x^{3/2} + C$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$$

$$\int X\, dx = \frac{1}{2} x\, X + \frac{a^2}{2} \ln|x + X| + C$$

$$\int xX\, dx = \frac{1}{3} X^3 + C$$

$$\int \frac{dx}{X} = \ln[x + X] + C$$

$$\int \frac{dx}{X^3} = \frac{1}{a^2}\frac{x}{X} + C$$

$$\int \frac{dx}{X^5} = \frac{1}{a^4}\left[\frac{x}{X} - \frac{1}{3}\frac{x^3}{X^3}\right] + C$$

$$\int \frac{x\, dx}{X} = X + C$$

$$\int \frac{x\, dx}{X^3} = -\frac{1}{X} + C$$

$$\int \frac{x\, dx}{X^5} = -\frac{1}{3X^3} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}(x/a) + C$$

$$\int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \tan^{-1}(x/a) + C$$

$$\int \frac{x\, dx}{a^2 + x^2} = \frac{1}{2} \ln|a^2 + x^2| + C$$

$$\int \frac{x\, dx}{(a^2 + x^2)^2} = -\frac{1}{2(a^2 + x^2)} + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln|(a + x)/(a - x)| + C = \frac{1}{a} \tanh^{-1}(x/a) + C$$

$$\int \frac{x\, dx}{(a^2 - x^2)} = -\frac{1}{2} \ln|a^2 - x^2| + C$$

$$\int \sin(ax)\, dx = -\frac{1}{a} \cos(ax) + C$$

$$\int \cos(ax)\, dx = \frac{1}{a} \sin(ax) + C$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq \pm b$$

$$\int \sin(ax)\cos(ax)\,dx = -\frac{\cos(2ax)}{4a} + C$$

$$\int \sin^n(ax)\cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{(n+1)a} + C, \quad n \neq -1$$

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln|\cos(ax)| + C$$

$$\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x + C$$

$$\int \cot^2(ax)\,dx = -\frac{1}{a}\cot(ax) - x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int b^{ax}\,dx = \frac{1}{a\ln(b)}b^{ax} + C$$

$$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$$

$$\int \ln(ax)\,dx = x\ln(ax) - x + C$$

$$\int x^n\ln(ax)\,dx = \frac{x^{n+1}}{n+1}\ln(ax) - \frac{x^{n+1}}{(n+1)^2} + C \qquad n \neq -1$$

$$\int \frac{1}{x}\ln(ax)\,dx = \frac{1}{2}[\ln(ax)]^2 + C$$

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax) + C$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax) + C$$
$$\int \tanh(ax)\,dx = \frac{1}{a}\ln[\cosh(ax)] + C$$
$$\int \coth(ax)\,dx = \frac{1}{a}\ln|\sinh(ax)| + C$$
$$\int \operatorname{sech}(ax)\,dx = \frac{1}{a}\sin^{-1}[\tanh(ax)] + C$$
$$\int \operatorname{csch}(ax)\,dx = \frac{1}{a}\ln|\tanh(ax/2)| + C$$
$$\int \sinh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} - \frac{x}{2} + C$$
$$\int \cosh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} + \frac{x}{2} + C$$
$$\int \tanh^2(ax) = x - \frac{1}{a}\tanh(ax) + C$$
$$\int \coth^2(ax)\,dx = x - \frac{1}{a}\coth(ax) + C$$
$$\int \operatorname{sech}^2(ax)\,dx = \frac{1}{a}\tanh(ax) + C$$
$$\int \operatorname{csch}^2(ax)\,dx = -\frac{1}{a}\coth(ax) + C$$

C.4 A partial list of definite integrals

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \qquad (a > 0)$$
$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2} \qquad (a > 0)$$
$$\int_0^\infty x^2e^{-ax}\,dx = \frac{2}{a^3} \qquad (a > 0)$$
$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} \qquad (a > 0, n > -1)$$
$$\int_0^\infty x^{1/2}e^{-ax}\,dx = \frac{1}{2a}\sqrt{\pi/a} \qquad (a > 0)$$
$$\int_0^\infty x^{-1/2}e^{-ax}\,dx = \sqrt{\pi/a} \qquad (a > 0)$$
$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2} \qquad (a > 0)$$
$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2} \qquad (a > 0)$$
$$\int_0^\infty xe^{-ax}\sin(bx)\,dx = \frac{2ab}{(a^2+b^2)^2} \qquad (a > 0)$$

Appendix C Useful mathematical tables

$$\int_0^{\infty} xe^{-ax}\cos(bx)\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \qquad (a>0)$$

$$\int_0^{2\pi} \sin(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \cos(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \sin^2(ax)\,dx = \pi \qquad (a=1,2,3,\ldots)$$

$$\int_0^{2\pi} \cos^2(ax)\,dx = \pi \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \cos(ax)\,dx = 0 \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin(ax)\,dx = \frac{1}{a}[1-\cos(a\pi)] \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin^2(ax)\,dx = \frac{\pi}{2} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \cos^2(ax)\,dx = \frac{\pi}{2} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi} \sin(ax)\sin(bx) = 0 \qquad a\neq b \text{ (} a \text{ and } b \text{ are integers)}$$

$$\int_0^{\pi} \cos(ax)\cos(bx) = 0 \qquad a\neq b \text{ (} a \text{ and } b \text{ are integers)}$$

$$\begin{aligned}\int_0^{\pi} \sin(ax)\cos(bx) &= 0 \qquad a = b \text{ (} a \text{ and } b \text{ are integers)}\\ &= 0 \qquad a\neq b \text{ but } (a+b) \text{ even}\\ &= \frac{2a}{a^2-b^2} \qquad a\neq b \text{ but } (a+b) \text{ odd}\end{aligned}$$

$$\int_0^{\pi/2} \sin(ax)\,dx = \frac{1}{a}[1-\cos(a\pi/2)]$$

$$\int_0^{\pi/2} \cos(ax)\,dx = \frac{1}{a}\sin(a\pi/2)$$

$$\int_0^{\pi/2} \sin^2(ax)\,dx = \frac{\pi}{4} \qquad (a=1,2,3,\ldots)$$

$$\int_0^{\pi/2} \cos^2(ax)\,dx = \frac{\pi}{4} \qquad (a=1,2,3,\ldots)$$

C.6 Some exact and approximate expressions for TEM waves in lossy media

	Exact	For good dielectric $\frac{\sigma}{\omega\varepsilon} << 1$	For good Conductor $\frac{\sigma}{\omega\varepsilon} >> 1$
Attenuation constant (Np/m)	$\alpha = Re\left(j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}\right)$	$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$
Phase constant (rad/m)	$\beta = Im\left(j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}\right)$	$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$
Intrinsic impedance (Ω)	$\hat{\eta} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\eta = \sqrt{\frac{\mu}{\varepsilon}}$	$\hat{\eta} = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}}$
Wavelength (m)	$\lambda = \frac{2\pi}{\beta}$	$\lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\lambda = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Wave velocity	$u_p = \frac{\omega}{\beta}$	$u_p = \frac{1}{\sqrt{\mu\varepsilon}}$	$u_p = \sqrt{\frac{2\omega}{\mu\sigma}}$
Skin depth (m)	$\delta_c = \frac{1}{\alpha}$	$\delta_c = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	$\delta_c = \sqrt{\frac{2}{\omega\mu\sigma}}$

C.7 Some physical constants

Constant	Symbol	Value
Velocity of light in vacuum	c	2.988×10^{8} m / s
Electronic charge (magnitude)	$\|e\|$	1.602×10^{-19} C
Electronic mass	m	9.109×10^{-31} kg
Electronic charge to mass ratio	$\|e\|/m$	1.759×10^{11} C / kg
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H / m
Permittivity of free space	ε_0	$8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$ F / m
Electron volt (energy)	$\|e\|V$	1.602×10^{-19} J
Boltzmann constant	k	1.381×10^{-23} J / K
Planck's constant	H	6.626×10^{-34} J. s